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Plane waves with weak singularities

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Abstract: We study a class of time dependent solutions of the vacuum Einstein equations which are plane waves with weak null singularities. This singularity is weak in the sense that though the tidal forces diverge at the singularity, the rate of divergence is such that the distortion suffered by a freely falling observer remains finite. Among such weak singular plane waves there is a sub-class which do not exhibit large back reaction in the presence of test scalar probes. String propagation in these backgrounds is smooth and there is a way but not unique to continue the metric beyond the singularity. This continued metric admits string propagation without the string becoming infinitely excited. We construct a one parameter family of smooth metrics which are at a finite distance in the space of metrics from the extended metric and a well defined operator in the string sigma model which resolves the singularity.

Keywords: Bosonic Strings, Conformal Field Models in String Theory, Penrose limit and pp-wave background.

Contents

1. Introduction

There has been a lot of interest in time dependent backgrounds in string theory. This has been mainly motivated by the desire to understand the cosmological singularity or the singularity behind horizons of black holes within string theory. These singularities are space like unlike the time like orbifold singularities which have been well understood in string theory. The simplest time dependent backgrounds which can be constructed in string theory are orbifolds involving either a boost or a null boost which have space like singularities and null singularities respectively at their fixed point [1]–[5]. Though these backgrounds are simple to construct and one can quantize the string modes they have been shown to be potentially unstable to formation of space like singularities like that of the ones behind horizons of black holes when probed by test particles. In [6] it was shown that the introduction of a single particle in the covering space of orbifolds by null boosts causes the space to collapse to a space like singularity, in [7] it was shown how a homogeneous energy distribution in the transverse directions causes the singularity in the orbifolds by null boosts to become space like. and in [8] it was argued that this instability was reflected in the divergence of string scattering amplitudes. Other time dependent backgrounds constructed involved orbifolding with boosts and a translation in an orthogonal direction [9] and orbifolding with null boosts and a translation [10, 8]. For the case or orbifolding with a boosts and a translation the singularity is time like and is resolved in string theory by orientifold planes [11] and they are shown to be stable to perturbations in [12]. Other simple time dependent backgrounds were studied in [13]–[16].

Perhaps the next simplest backgrounds involving null singularities in string theory are exact plane waves with null singularities. Their metric is given by

$$ds^2 = dudv + F_{ij}(u)x^i x^j du^2 + dx^i dx^i \,. \tag{1.1}$$

If F_{ij} is traceless they are solutions to the vacuum Einstein equations and for $\mathrm{Tr}F \neq 0$ they are solutions in string theory if they are supported by appropriate antisymmetric tensor-form field strengths or a non constant dilaton. These exact plane wave backgrounds are singular from the general relativity point of view if F_{ij} diverges at some point $u = u_0$ (see for instance in [17, 18]) String propagation on these backgrounds were explored in [19]–[23] and more recently, [24, 25] have investigated a plane wave in which the classical string equations of motion can be solved explicitly.

As plane waves with null singularities are not obtained by an orbifold of Minkowski space they do not posses the instability found in [6] for the case of oribolds involving boosts and null boosts. They are exact solutions of string theory to all orders in α' [20, 21]. The presence of the covariantly constant null Killing vector ensures absence of particle production [26] in these backgrounds and that they preserve half the supersymmetry when embedded in super string theory. This makes them good candidates to study the issue of null singularities in string theory though in general the string modes cannot be quantized explicitly.

In this paper we study a class of plane waves which are solutions of the vacuum Einstein equations with a weak null singularity. They are weak in the sense defined in [27, 28], the tidal forces diverge at the singularity but the distortion caused by them on an object freely falling into the singularity remains finite. Among these weak singularities we find a sub-class in which the back reaction on a test scalar probe remains finite at the singularity. This and the fact that the singularity is weak ensures that classical string modes and their first derivative remains finite at the singularity. Thus there is a way to continue them across the singularity by matching the modes and their first derivative across the singularity to extend the metric beyond it. This extension of the metric is not unique.

In string theory, for a plane wave to be non-singular not only should the string modes be extended across the singularity but also the the string should not become infinitely excited as it passes through the singularity [21, 22]. For instance in the case of exact plane waves which are vacuum solutions of Einstein's equations with a delta function singularity (eg. $F_{ij} = \epsilon_{ij}\delta(u)$) the string modes can be continued across the singularity but the string gets infinitely excited as it passes through it and therefore this plane wave is singular from the point of view of string theory. In the case of weak singular plane waves which are studied in this paper we show that the string modes continued across the singularity are not infinitely excited, thus justifying the extended metric. As further support for the extended metric we show that there exists a family of smooth metrics which are at a finite distance in the space of metrics from the extended metric. We demonstrate the existence of a deformation which smoothes out the singularity.

The paper is organized as follows. In section 2 we introduce the backgrounds we will study in this paper and review the notion of a weak singularity. We show that among weak singular plane waves there is a subset for which the energy of a test scalar particle

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does not diverge at the singularity ensuring that the back reaction is mild. In section 3 we study string propagation in this background, we show that the classical string modes and their first derivative is finite at the singularity, this allows one to smoothly continue them across the singularity. In section 4 we show that the string does not get infinitely excited as it passes through the singularity in the extended metric. Then we show that there are smooth metrics which are a finite distance in the moduli space from the extended metric and the existence of a well defined operator in the string sigma model which resolves the singularity. In section 5 we state our conclusions.

2. Plane waves with weak singularities

In this section we review the conditions of when a plane wave is singular and demonstrate when they have weak singularities. We then show that among the class of weakly singular plane waves there is a subset in which the stress energy tensor of a test scalar does not diverge at the singularity.

2.1 A singular plane wave

For the purposes of this paper we restrict our attention to the following metric

$$ds^2 = dudv + F(u)(x^2 - y^2)du^2 + dx^2 + dy^2\,, \tag{2.1}$$

these are solutions to the vacuum Einstein equations. The considerations in this paper can be generalized to other plane wave solutions. $F(u)$ in (2.1) is any function which vanishes at infinity and singular at $u = 0$. For example $F(u) = e^{-u}/u^\alpha$ with $\alpha > 0$ satisfies this criteria. The metric is defined for $u > 0$, the geodesic equations for this metric is given by

$$\begin{aligned}
\frac{d^2u}{d\lambda^2} &= 0\,, \\
\frac{d^2x}{d\lambda^2} - F(u)x\left(\frac{du}{d\lambda}\right)^2 &= 0\,, \\
\frac{d^2y}{d\lambda^2} + F(u)y\left(\frac{du}{d\lambda}\right)^2 &= 0\,, \\
\frac{d}{d\lambda}\left(\frac{du}{d\lambda}\frac{dv}{d\lambda} + \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + F(u)(x^2 - y^2)\left(\frac{du}{d\lambda}\right)^2\right) &= 0\,,
\end{aligned} \tag{2.2}$$

where λ refers to the affine parameter. It is clear that from these equations that if there is a singularity in $F(u)$ at $u = 0$ the space time exists only for $u > 0$ as every time like geodesic with $u = -p\lambda$, $\lambda \in (-\infty, 0)$, reaches the singularity at $\lambda = 0$. The point $u = 0$ is not a scalar curvature singularity as all curvature invariants vanish for the plane wave metric. One can characterize the singular behaviour of the curvature tensor in a coordinate invariant way as follows. Consider a time like geodesic which ends at the singularity, an orthonormal frame which is parallel propagated along this geodesic is given by

$$\begin{aligned}
e_0 &= (p, \dot{v}, \dot{x}, \dot{y})\,, & e_1 &= \left(p, \dot{v} + \frac{2}{p}, \dot{x}, \dot{y}\right), \\
e_2 &= \left(0, -\frac{2\dot{x}}{p}, 1, 0\right), & e_3 &= \left(0, -\frac{2\dot{y}}{p}, 0, 1\right).
\end{aligned} \tag{2.3}$$

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Here p is the velocity for the u coordinate, the vector e_0 is the tangent to the geodesic and normalized as $e_0^\mu e_{0\mu} = -1$, the dot refers to derivative with respect to the affine parameter λ. The components of the Riemann curvature with respect to this frame diverge at the singularity, they are given by

$$\begin{aligned} R_{2121} &= -p^2 F\,, \qquad & R_{3232} &= p^2 F\,, \\ R_{2020} &= -p^2 F\,, \qquad & R_{3030} &= p^2 F\,. \end{aligned} \tag{2.4}$$

2.2 Condition for a plane wave with a weak singularity

When a body falls into this singularity the tidal forces diverge. The tidal forces suffered by an object falling into the singularity in its parallel propagated frame are given by the curvature components in the time like directions in (2.4). The singularity is weak if the rate of divergence is small enough so that the object is not distorted as it hits the singularity. The distortion is proportional to the second integral of the tidal force [27]. Consider the case of power law divergence at the origin $F(u) = 1/u^\alpha$, the distortion is finite at the origin if $0 < \alpha < 2$

This notion was made more precise by [28]. A singularity is defined to be weak if for every time like geodesic which ends on it there exists linearly independent space like vorticity-free Jacobi fields along the geodesic normal to its tangent vector which define a volume element that remains finite at the singularity. This captures the intuitive notion of the distortion of the object being finite. We now verify that for the case of power law divergence $F(u) = 1/u^\alpha$ with $0 < \alpha < 2$ the Jacobi fields define a finite volume element at the singularity.

Jacobi fields satisfy the geodesic deviation equations, in the orthonormal frame (2.3) the space like Jacobi fields satisfy

$$\begin{aligned} \frac{d^2\eta^1}{d\lambda^2} &= 0\,, \\ \frac{d^2\eta^2}{d\lambda^2} - p^2\frac{1}{u^\alpha}\eta^2 &= 0\,, \\ \frac{d^2\eta^3}{d\lambda^2} + p^2\frac{1}{u^\alpha}\eta^3 &= 0\,, \end{aligned} \tag{2.5}$$

here $u = p\lambda$. The geodesic deviation equation is diagonal, that is it does not mix the space like Jacobi fields η^i among each other, therefore the vorticity of these Jacobi fields is zero. Solutions of these equations for $0 < \alpha < 2$ are known in closed form. Consider the time like geodesic with $u = p\lambda$, an example of Jacobi fields which remain finite at the origin is given by

$$\eta^1 = \lambda + 1\,, \qquad \eta^2 = \sqrt{p\lambda}K_\nu\left(2\nu(p\lambda)^{\frac{1}{2\nu}}\right), \qquad \eta^3 = \sqrt{p\lambda}Y_\nu\left(2\nu(p\lambda)^{\frac{1}{2\nu}}\right), \tag{2.6}$$

where $\nu = 1/(2-\alpha)$ and Y_ν and K_ν are the Bessel function and the modified Bessel function respectively. From (A.6), (A.7) and (A.8), it is clear that these functions are finite at the origin. Thus the volume defined by the product $\eta^1\eta^2\eta^3$ is finite at the origin. Therefore for power law divergences $F(u) = 1/u^\alpha$, the singularity is weak if $\alpha < 2$. Note that not only the volume defined by the Jacobi fields is finite for these class of singularities but also each of the Jacobi field is finite at the singularity.

2.3 Scalars fields in a weakly singular plane wave

We now show that among the class of weak singularities discussed in the previous section we can find a subclass for which the stress energy tensor of a massless scalar does not diverge as seen by any time like observer falling in to the singularity. The evaluation of the stress energy tensor in the frame of the in falling time like observer provides an invariant method of testing whether the scalar field will cause an infinite back reaction at the singularity or not. This method of testing for back reaction of a scalar in the presence of null singularities is common in the literature, see for instance in [29].

It is convenient to work in the Rosen coordinate system which are well behaved at the singularity. The details of the various Rosen coordinate systems for the plane wave with $F(u) = 1/u^{\alpha}$ and $0 < \alpha < 2$ are given in the appendix. In this coordinates the metric for the plane wave is given by

$$ds^2 = d\mathrm{u}d\mathrm{v} + f^2(\mathrm{u})d\mathrm{x}^2 + g^2(\mathrm{u})d\mathrm{y}^2, \tag{2.7}$$

where f and g are

$$f(\mathrm{u}) = \sqrt{\mathrm{u}}K_{\nu}(2\nu\mathrm{u}^{\frac{1}{2\nu}}), \qquad g(\mathrm{u}) = \sqrt{\mathrm{u}}Y_{\nu}(2\nu\mathrm{u}^{\frac{1}{2\nu}}). \tag{2.8}$$

These Rosen coordinates are well behaved at the singularity $u = 0$, from the expansions in (A.6), (A.7) and (A.8), it is clear that they are finite at the origin. The massless scalar wave equation in the background (2.7) reduces to

$$\frac{1}{fg}\left(\partial_{\mathrm{u}}(2fg\partial_{\mathrm{v}}\phi) + \partial_{\mathrm{v}}(2fg\partial_{\mathrm{u}}\phi) + \partial_{\mathrm{x}}\left(\frac{g}{f}\partial_{\mathrm{x}}\phi\right) + \partial_{\mathrm{y}}\left(\frac{f}{g}\partial_{\mathrm{y}}\phi\right)\right) = 0\,. \tag{2.9}$$

The above equations are invariant under translations in $\mathrm{v}, \mathrm{x}, \mathrm{y}$, therefore we can solve these equations by assuming the form $\phi = \phi(\mathrm{u})\exp(i\omega v + ik_x x + ik_y y)$. Substitution this form in (2.9) we obtain an ordinary equation for $\phi(u)$. The complete solution for ϕ is given by

$$\phi(\mathrm{u}, w, k_x, k_y) = \frac{1}{\sqrt{fg}}\exp\left(i\omega\mathrm{v} + ik_x\mathrm{x} + ik_y\mathrm{y} + \frac{1}{4i\omega}\int d\mathrm{u}\left(\frac{k_x^2}{f^2} + \frac{k_y^2}{g^2}\right)\right). \tag{2.10}$$

Note that the wave function is well defined and smooth at the origin. The nontrivial dependence of this wave function is in the u coordinate. The derivative of the wave function with respect to u is given by

$$\partial_{\mathrm{u}}\phi = \left(-\frac{1}{2}\frac{\partial_{\mathrm{u}}(fg)}{fg} + \frac{1}{4iw}\left(\frac{k_x^2}{f^2} + \frac{k_y^2}{g^2}\right)\right)\phi\,. \tag{2.11}$$

As f and g are well behaved at the origin the behaviour of the derivative at the origin is dictated by the of logarithmic derivative of the product fg at the origin. From the expansions in (A.5), (A.6), (A.7) and (A.8) it is clear that this derivative is finite only for $0 < \alpha < 1$. The stress energy tensor is a bilinear function of the derivatives of the scalar field, therefore we expect the stress energy tensor evaluated in the frame of the in falling observer will be well behaved for $0 < \alpha < 1$.

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The general solutions for the scalar field is given by

$$\phi(\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y}) = \int dw dk_x dk_y \phi(w,k_x,d_y)\chi(w,k_x,k_y)\,. \tag{2.12}$$

Here $\chi(w,k_x,k_y)$ are the Fourier coefficients which are determined by boundary conditions given on a constant u surface. If the boundary conditions are given in a patch which is beyond the validity of the Rosen coordinates (2.7), then by solving the wave equation in that patch corresponding to the boundary conditions and using the overlap of patches one can determine $\chi(w,k_x,k_y)$.[1] For simplicity in our analysis we will assume our boundary conditions are such that $\chi(w',k_x,k_y) = 1/2(\delta(w-w')+\delta(w+w'))\delta(k_x)\delta(k_y)$. This is basically assuming the scalar field is independent in x and y and it is a cosine wave with frequency w. The analysis which we detail below can be easily repeated for an arbitrary Fourier coefficient $\chi(w,k_x,k_y)$ The non-zero components of the stress energy tensor of the cosine wave $\phi(\mathrm{u},\mathrm{v}) = \frac{1}{\sqrt{fg}}\cos(w\mathrm{v})$ are given by

$$\begin{aligned}
T_{\mathrm{uu}} &= \left(\frac{\partial_{\mathrm{u}}(fg)}{fg}\right)^2 \cos^2(w\mathrm{v})\,,\\
T_{\mathrm{vv}} &= w^2\sin^2(w\mathrm{v})\,,\\
T_{\mathrm{xx}} &= \frac{f^2 w\partial_{\mathrm{u}}(fg)}{2fg}\cos(w\mathrm{v})\sin(w\mathrm{v})\,,\\
T_{\mathrm{yy}} &= \frac{f^2 w\partial_{\mathrm{u}}(fg)}{2fg}\cos(w\mathrm{v})\sin(w\mathrm{v})\,.
\end{aligned} \tag{2.13}$$

All the components are well behaved at the origin if $0<\alpha<1$. The stress energy tensor for the sine wave is obtained by replacing the cosines by the sines and vice-versa in the above formulae.

One might suspect that the reason that the stress energy tensor is well behaved is due to the choice of nice coordinates near the singularity and a coordinate artifact. In fact if the stress energy tensor given in (2.13) is converted to the plane wave coordinates in (2.1) the stress energy tensor is divergent. The natural way to check if the stress energy tensor is divergent in a coordinate invariant manner is to evaluate the components of the stress energy tensor as seen in the frame of an observer traveling on any time like geodesic [29]. We now show that that the stress energy tensor of a scalar field seen by a in falling observer is finite in the class of singular plane waves with $F(u) = 1/u^\alpha$ with $0<\alpha<1$. The geodesic equations in Rosen coordinates are given by

$$\begin{aligned}
\frac{d^2\mathrm{u}}{d\lambda^2} &= 0\,,\\
\frac{d^2\mathrm{x}}{d\lambda^2} + 2\frac{\dot f}{f}\frac{d\mathrm{u}}{d\lambda}\frac{d\mathrm{x}}{d\lambda} &= 0\,,\\
\frac{d^2\mathrm{y}}{d\lambda^2} + 2\frac{\dot g}{g}\frac{d\mathrm{u}}{d\lambda}\frac{d\mathrm{y}}{d\lambda} &= 0\,,\\
\frac{d}{d\lambda}\left(\frac{d\mathrm{v}}{d\lambda} + f^2\left(\frac{d\mathrm{x}}{d\lambda}\right)^2 + g^2\left(\frac{d\mathrm{y}}{d\lambda}\right)^2\right) &= 0\,.
\end{aligned} \tag{2.14}$$

[1] One might have to go through several Rosen coordinate patches.

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The solutions for the tangent vector along the geodesic is given by

$$\frac{d\mathrm{u}}{d\lambda} = 1\,, \qquad \frac{d\mathrm{x}}{d\lambda} = \frac{a}{f^2}\,, \qquad \frac{d\mathrm{y}}{d\lambda} = \frac{b}{g^2}\,,$$
$$\frac{d\mathrm{v}}{d\lambda} = -1 - \frac{a^2}{f^2} - \frac{b^2}{g^2}\,. \tag{2.15}$$

Here we have scaled λ so that velocity in the u direction is unity and normalized the tangent vector such that its norm is -1. a and b are arbitrary constants. Note that the velocity along the v direction is finite in the Rosen coordinates unlike the case in the plane wave coordinates as in (2.2). Let the tangent vector along the geodesic be denoted by v_0^μ. The other vectors in the parallel propagated orthonormal tetrad along this geodesic are

$$\begin{aligned} v_1 &= \left(1,\ 1 - \frac{a^2}{f^2} - \frac{b^2}{g^2},\ \frac{a}{f^2},\ \frac{b}{g^2}\right), \\ v_2 &= \left(0, -\frac{2a}{f}, \frac{1}{f}, 0\right), \\ v_3 &= \left(0, -\frac{2b}{g}, 0, \frac{1}{g}\right). \end{aligned} \tag{2.16}$$

It is clear that since all the components of this tetrad are well behaved at $\mathrm{u} = 0$ the stress energy tensor in this frame will be well behaved. For instance the 00 component of the stress energy tensor of the scalar field in the frame of the in falling observer is given by

$$\begin{aligned} T_{00} &= T_{\mu\nu} v_0^\mu v_0^\nu\,, \\ &= T_{\mathrm{uu}} + T_{\mathrm{vv}} \left(\frac{d\mathrm{v}}{d\tau}\right)^2 + T_{\mathrm{xx}} \left(\frac{d\mathrm{x}}{d\tau}\right)^2 + T_{\mathrm{yy}} \left(\frac{d\mathrm{y}}{d\tau}\right)^2. \end{aligned} \tag{2.17}$$

It is clear from (2.13) and from (2.15) the above expression for T_{00} is well behaved at $\mathrm{u} = 0$. Similarly from (2.13) and (2.16) the other components also will be well behaved at the singularity. The apparent divergence of the stress energy tensor in the plane wave coordinate given in (2.1) is canceled when contracted with the velocity vectors of the geodesic. Note that in (2.2) the velocity vector along the v direction diverges.

Thus for the subclass of plane wave singularities which diverge as $F(u) = 1/u^\alpha$ with $0 < \alpha < 1$ the stress energy tensor of a scalar field as seen by an in falling geodesic does not diverge, therefore these backgrounds are classically stable from large back reaction effects for a scalar probe.

3. Strings in weak singular plane waves

In this section we study string propagation in plane waves with weak singularities. We show that that the classical string modes for plane waves with singularity of the form $F(u) = 1/u^\alpha$ and $0 < \alpha < 1$ are well behaved at the singularity, in fact even the derivative of the string modes are well behaved. This suggests a natural method for extension of the metric in (2.1) beyond the singularity $u = 0$. Considering the metric as $F(u) = 1/|u|^\alpha$ One can match the classical string modes and its derivative at $u = 0$ and continue the modes smoothly across $u = 0$. We discuss this continuation of the string modes in this section and will provide more justification for this extension of the metric in the section 4.

Consider the metric given in (2.1) embedded in bosonic string theory.[2] The world sheet action of the bosonic string is given by

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\Big(\partial_a U\partial^a V + F(U)(X^2-Y^2)\partial_a U\partial^a U + \\ + \partial_a X\partial^a X + \partial_a Y\partial^a Y + \partial_a X^i\partial^a X_i\Big)\,. \qquad (3.1)$$

Here $i = 3,\dots 25$, we have used the Minkowski signature for the world sheet metric. The classical string modes can be studied in the light cone gauge, substituting $U = p\tau$ in the action (3.1) we obtain

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\bigg(-p\frac{dV}{d\tau} - p^2F(p\tau)(X^2-Y^2) + \\ + \partial_a X\partial^a X + \partial_a Y\partial^a Y + \partial_a X^i\partial^a X_i\bigg)\,. \qquad (3.2)$$

The coordinates X^i are free and their solution is given by the usual mode expansion, we will ignore these modes in our discussion. The non-trivial equations of motion are for the X and Y fields, they are given by

$$\begin{aligned}\partial^a\partial_a X + p^2F(p\tau)X &= 0\,,\\ \partial^a\partial_a Y - p^2F(p\tau)Y &= 0\,.\end{aligned} \qquad (3.3)$$

Translations in the world sheet coordinate σ is a symmetry of the above equations, this and the fact that σ is a periodic coordinate imply that we can expand the modes as

$$X(\tau,\sigma) = \sum_{n=-\infty}^{n=\infty} X_n(\tau)e^{in\sigma}\,, \qquad Y(\tau,\sigma) = \sum_{n=-\infty}^{n=\infty} Y_n(\tau)e^{in\sigma} \qquad (3.4)$$

where X_n and Y_n satisfy the differential equations

$$\begin{aligned}\ddot{X}_n + n^2X_n - p^2F(p\tau)X_n &= 0\,,\\ \ddot{Y}_n + n^2Y_n + p^2F(p\tau)Y_n &= 0\,.\end{aligned} \qquad (3.5)$$

Thus the equations for the classical modes reduce to equations for time dependent oscillators with repulsive and attractive potentials. The classical modes for the coordinate V is determined by the following constraint equations

$$\begin{aligned}-p\dot{V} &= (\dot{X})^2 + (\dot{Y})^2 + (X')^2 + (Y')^2 + F(p\tau)(X^2-Y^2)\,,\\ -pV' &= 2(\dot{X}X' + \dot{Y}Y')\,,\end{aligned} \qquad (3.6)$$

here the derivatives are with respect to the world sheet coordinates.

Let us now discuss the modes for singular plane waves with $F(u) = 1/u^\alpha$ and $0 < \alpha < 1$. The zero modes for the X and Y coordinates are the geodesic equations for these coordinates in (2.2), their solutions are given by

$$\begin{aligned}X_0 &= \alpha_0\sqrt{p\tau}I_\nu\left(2\nu(p\tau)^{1/2\nu}\right) + \tilde{\alpha}_0\sqrt{p\tau}K_\nu\left(2\nu(p\tau)^{1/2\nu}\right)\,,\\ Y_0 &= \beta_0\sqrt{p\tau}J_\nu\left(2\nu(p\tau)^{1/2\nu}\right) + \tilde{\beta}_0\sqrt{p\tau}Y_\nu\left(2\nu(p\tau)^{1/2\nu}\right)\,,\end{aligned} \qquad (3.7)$$

[2]This discussion can be easily extended for the case of the superstring.

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where $\nu = 1/(2-\alpha)$ and $\alpha_0, \tilde{\alpha}_0, \beta_0, \tilde{\beta}_0$ are arbitrary constants. J_ν, Y_ν and I_ν, K_ν are the Bessel functions and modified Bessel functions respectively. Note that for $0 < \alpha < 1$, ν always lies between 0 and 1, and therefore ν never takes on integral values. From the expansion of these Bessel functions given in (A.5) and (A.6) it is easy to see that they are finite and their derivatives are also finite at the origin for $0 < \alpha < 1$.

Consider the non zero modes for the coordinate X_n, the behaviour of these modes can at $u = 0$ can be studied by substituting $X_n = x_n h$ in (3.5) with $h(p\tau) = \sqrt{p\tau} I_{-\nu}\left(2\nu (p\tau)^{1/2\nu}\right)$ the equations of motion for X_n then reduces to

$$\ddot{x}_n + 2\dot{x}_n \frac{1}{h(p\tau)} \frac{d}{d\tau} h(p\tau) + n^2 x_n = 0 \,, \tag{3.8}$$

where we have used the fact that h solves the equation

$$\ddot{h} + \frac{1}{\tau^\alpha} h = 0 \,. \tag{3.9}$$

The expansion for h is given by

$$h(\tau) = \nu^{-\nu} \left(\frac{1}{\Gamma(-\nu+1)} + \frac{\nu^2 \tau^{2-\alpha}}{1!\Gamma(-\nu+2)} + \frac{\nu^4 \tau^{2(2-\alpha)}}{2!\Gamma(-\nu+3)} + \cdots \right) , \tag{3.10}$$

it is clear from the above expansion that the logarithmic derivative of h goes to zero at $\tau = 0$ if $0 < \alpha < 1$. Therefore from (3.8) the two linearly independent solutions for X_n as $\tau \to 0$ are given by

$$X_n = e^{\pm in\tau} \sqrt{p\tau} I_{-\nu}(2\nu (p\tau)^{\frac{1}{2\nu}}) \left(1 + O(\tau^{2-\alpha})\right) , \qquad \text{as } \tau \to 0 \,. \tag{3.11}$$

The fact that these solutions are linearly independent can be shown by the evaluation of the Wronskian. Thus the modes are well behaved at $\tau = 0$ furthermore from (3.11) it is also clear that the derivatives are also well behaved at the origin. A similar argument for the modes Y_n shows that they are given by

$$Y_n = e^{\pm in\tau} \sqrt{p\tau} J_{-\nu}(2\nu (p\tau)^{\frac{1}{2\nu}}) \left(1 + O(\tau^{2-\alpha})\right) , \qquad \text{as } \tau \to 0 \,. \tag{3.12}$$

Thus the modes for the Y_n and its derivative are also well behaved at the origin. We have also checked the the fact that these modes are well behaved at the origin by a Frobenius expansion which can be performed for rational values of α and numerically. Note that the in determining the behaviour of the modes the crucial role was played by the change of variable from X_n to x_n. This transformation is precisely the transformation to a specific Rosen coordinates as can be seen from (A.1). The modes of the coordinate V is determined from the constraints (3.6), it is easy to see from these equations that V is finite but $\dot{V}$ in general diverges at the origin.

We now outline a natural but not unique completion of the plane wave metric with a power law divergence in $F(u) = 1/u^\alpha$ and $0 < \alpha < 1$ to the region $u < 0$. Consider the extension to be the plane wave metric with $F(u) = 1/|u|^\alpha$ at the origin, then if the two linearly independent solutions for X_n are $X_n^{(1)}(\tau)$ and $X_n^{(2)}(\tau)$ for $\tau > 0$, the two linear independent solutions for $\tau < 0$ are given by $X_n^{(1)}(-\tau)$ and $X_n^{(2)}(-\tau)$. We have seen that for $0 < \alpha < 1$ these modes and their derivatives are well behaved at the origin, therefore one can match the value of the modes and their derivative for the solutions in the two regions at the origin and naturally continue them across the singularity. This procedure cannot be done for the case of singularities with $1 \leq \alpha < 2$ as the derivative of the modes diverge at the origin. The modes for the world sheet coordinate V is determined from the constraints given in (3.6). Note that in this completion of the metric the zero modes as well as the non-zero modes of the string are in equal footing, unlike the case of the usual orbifolds in which geodesics or the zero modes cannot be completed while the non-zero modes of the string can be extended.[3] Certainly it is clear that this procedure of completion of the metric is not unique, one might choose to complete it with any other non-singular metric for $u < 0$. In the next section we will study two requirements for the extended metric arising from string theory we see that these requirements still do not fix the extension to be unique.

4. Tests for the extended metric

We have seen in the previous section that for plane waves with singularity $F(u) = 1/u^\alpha$ that there is a natural extension of the metric to $u < 0$ with $F(u) = 1/|u|^\alpha$. One can ensure that all the string modes are continuous and differentiable at the origin for $0 < \alpha < 1$. This still does not guarantee that the metric is singularity free from the point of view of string theory. Though plane wave metrics do not exhibit particle creation there is a phenomenon of mode creation, a string passing through the singularity can become excited, for the metric to be non-singular from the point of view of string theory it is important to show that mode creation is finite. For instance with $F(u) = \delta(u)$ though the modes can be continued across the singularity (the derivative is discontinuous) the string gets infinitely excited as it passes through this singularity. This infinte excitation is infact due to the infinite extent of the shock wave profile and it similar to the case of $\alpha = 1$ [23, 30]

In this section we show that the mode creation in the continued metric with $F(u) = 1/|u|^\alpha$ and $0 < \alpha < 1$ is finite, this was noticed in [23]. The extension of the metric to $u < 0$ will also be natural if there exists a family of smooth metrics which are at a finite distance from the metric $F(u) = 1/|u|^\alpha$ in the space of metrics. Then this extended metric is just a point in the moduli space, much like the case of the R^4/Z_2 orbifold, the singular R^4/Z_2 metric is just a limit of the smooth Eguchi-Hansen space. We construct such a family of smooth metrics for the extended metric. We also show that there is a well defined operator in the string sigma model which resolves the singularity. Thus this class of plane wave metrics can be smoothened out much like the R^4/Z_2 orbifold singularity.

4.1 Mode creation

The world sheet theory for the string in these time dependent backgrounds has time dependent potentials. Time dependent potentials in general will mix positive frequencies and negative frequencies on the world sheet, which implies there is a transition amplitude

[3]The author thanks Ashoke Sen for pointing this out.

between the oscillator modes of the vacuum at $\tau = -\infty$ to the oscillator modes of the vacuum at $\tau = -\infty$, here τ stands for the world sheet time. The mixing of modes will in general excite string modes as it passes through the time dependent background. To compute the amplitude for mode creation through a time dependent back ground one needs to know the solution for the differential equations (3.5). Solutions for these equations do not exist in closed form,[4] we therefore resort to approximate methods. Since we are interested in the behaviour close to the singularity our methods should be valid in that region. It is easy to see that for $\alpha < 2$ the WKB method is not valid close to $u = 0$, the other approximate method is perturbation theory. In [23] the (mass)^2 of the excited modes for the case strings passing through plane waves singularities with $0 < \alpha < 1$ was estimated using first order perturbation theory and show to be finite. Here we emphasize the validity doing perturbation theory in spite of the presence of the singularity at $u = 0$ We estimate the mode creation to the second order in perturbation theory and show that string does not get infinitely excited as it passes through the singularity.

As we are only interested in the singularity at the origin we can assume that the function $F(u) = \frac{1}{|u|^\alpha}$ is modulated by a function which is smooth and falls of at infinity. To keep the calculations involved simple we perform the analysis below with $F(u) = 1/|u|^\alpha$ for $|u| < T$ and $F(u) = 0$ for $|T| > 0$ sufficiently large but finite. We set up the perturbation theory for the modes of the coordinate X. A similar treatment will go through for the coordinate Y. The solution of equations of motions for the world sheet coordinate X is given in terms of the following series

$$\begin{aligned} X_n(\tau) &= X_n^{(0)}(\tau) + \int d\tau_1 G(\tau - \tau_1) p^2 F(p\tau_1) X_n^{(0)}(\tau_1) + \\ &+ \int d\tau_2 d\tau_1 G(\tau - \tau_2) p^2 F(p\tau_2) G(\tau_2 - \tau_1) p^2 F(p\tau_1) X_n^{(0)}(\tau_1) + \cdots . \end{aligned} \tag{4.1}$$

All the integrals run from $-\infty$ to ∞. The expansion parameter can be thought of as the light cone momentum p. $X_n^{(0)}$ is the solution of the unperturbed equation

$$\ddot{X}_n^{(0)} + n^2 X_n^{(0)} = 0\,, \tag{4.2}$$

and $G(\tau - \tau')$ is its Green's function which is given by

$$G(\tau - \tau') = \frac{\theta(\tau' - \tau)}{2in}\left(e^{in(\tau-\tau')} - e^{-in(\tau-\tau')}\right), \tag{4.3}$$

where $\theta(\tau - \tau')$ is the step function which ensures that the above Green's function vanishes for $\tau > \tau'$. We require this boundary condition as we are propagating the solutions from $\tau = \infty$ to $\tau = -\infty$. It is clear from the perturbative expansion in (4.1) all the integrals are well defined if $F(u) = 1/|u|^\alpha$ with $0 < \alpha < 1$. This is because the interaction potential $F(p\tau)$ always occurs with an integration in τ, so the integrals around $u = 0$ are well defined.[5] Note that it is not clear that one can trust the perturbative analysis for $\alpha \geq 1$.

[4]For the case $\alpha = 1$ solutions exist in closed form.

[5]For $F(u) = \delta(u)$ the perturbation expansion truncates at the first order giving the exact answer.

For mode creation what is important is to obtain the amplitude of the incoming wave after scattering from the potential. Consider the incoming wave at $\tau = \infty$ as $e^{in\tau}$, then the amplitude of the coefficient of the outgoing wave $e^{-in\tau}$ at $\tau = -\infty$ is given by

$$B_n e^{-in\tau} = \lim_{\tau \to -\infty} \int d\tau' \left(\tilde{G}(\tau - \tau') - \frac{1}{in}\frac{d}{d\tau}\tilde{G}(\tau - \tau') \right) V(\tau') e^{in\tau'} . \tag{4.4}$$

Here B_n refers to the coefficient of the outgoing wave and $\tilde{G}$ stands for full Green's function whose expansion is given by

$$\tilde{G}(\tau - \tau') = G(\tau - \tau') + \int d\tau_1 G(\tau - \tau_1) p^2 F(p\tau_1) G(\tau - \tau') + \cdots . \tag{4.5}$$

Evaluating B_n to the first order in perturbation theory gives

$$B_n^{(1)} = \frac{\hat{F}(-2n)}{2in} . \tag{4.6}$$

Here $\hat{F}$ is the Fourier transform of the potential defined by $\hat{F}(w) = \int dt p^2 F(pt) e^{iwt}$. For the potential $F(u) = 1/u^\alpha$ with $0 < \alpha < 1$ it is given by

$$\hat{F}(w) = 2\frac{p^{2-\alpha}}{|w|^{1-\alpha}} \Gamma(1-\alpha) \sin\left(\frac{\pi\alpha}{2}\right) + O\left(\frac{1}{wT^\alpha}\right) , \tag{4.7}$$

where the $O(1/wT^\alpha)$ terms refers to the corrections due to the fact that the potential is non zero only for $|u| < T$. From the Bogoliubov transformations for the oscillator modes the expectation value of the number operator at $\tau = -\infty$ of the vacuum at $\tau = \infty$ is given by

$$\langle 0_+ | N_n | 0_+ \rangle = |B_n|^2 , \tag{4.8}$$

where N_n refers to the number operator of either the left movers or the right movers in the nth level. The expectation value of the number operator can be translated to the expectation value of the mass of the excited string after it passes through the singularity which is given by

$$\langle M^2 \rangle = \frac{4}{\alpha'} \left(2 \sum_n n \langle N_n^i \rangle - 2 \right) . \tag{4.9}$$

From the equations (4.6), (4.7) and (4.8) we see the mass contribution from the excited modes corresponding to the X coordinate in first order perturbation theory is given by

$$\langle M_x^2 \rangle \sim \frac{2}{\alpha'} \sum_n \frac{1}{n^{3-2\alpha}} . \tag{4.10}$$

It is easy to see that this sum converges for $0 < \alpha < 1$. A similar analysis holds for the Y coordinate, thus the (mass)^2 of the excited string remains finite in first order perturbation theory. To show that the leading estimate for the amplitude for mode creation is given by the first order term we evaluate the Bogoliubov coefficient B_n to the next order in perturbation theory. This is given by

$$\begin{aligned} B_n^{(2)} &= \frac{1}{2n^2} \left(\hat{F}(-2n)\hat{F}(0) - \hat{F}(-4n)\hat{F}(-2n) \right) , \\ &= \frac{p^{2(2-\alpha)}}{2n^2(n^{1-\alpha})} \left[\frac{2T^{1-\alpha}}{1-\alpha} - \left(\frac{1}{4n}\right)^{1-\alpha} \right] + O\left(\frac{1}{nT^\alpha}\right) , \end{aligned} \tag{4.11}$$

note that $B_n^{(2)}$ is suppressed by higher powers of n and is finite for a given T, therefore the amplitude for mode creation given in (4.10) is the leading estimate.

4.2 Nonsingular metrics close to the extended metric

In this section we show that there exists a one parameter family of smooth metrics which are at a finite distance from plane wave metrics given in (2.1) with a singularity of the form $F(u) = 1/|u|^\alpha$ with $0 < \alpha < 1$ at the origin. The natural way to find a smooth metric which reduces to this singular metric in a limit is to deform the function $F(u)$ to any smooth function at $u = 0$. For instance the function $F(u) = (u^2 + a^2)^{-\alpha/2}$ or $F(u) = (|u| + a)^{-\alpha}$, $a > 0$[6] are possible candidates. The difficulty in considering these kind deformations directly in the plane wave coordinates or (2.1) is that they are all null deformations, that is their distance in the space of metrics is null. This is easily seen as follows, the distance in the space of metrics is given by

$$\Delta = \int d^4x \sqrt{g} g^{\mu\nu} g^{\rho\sigma} \delta g_{\mu\rho} \delta g_{\nu\sigma} \,, \tag{4.12}$$

where $\delta g_{\mu\nu}$ refers to the change in the metric. Note that the metric we have in (2.1) is such that $g^{\mu\nu}$ is zero if μ or ν refers to the u direction. Therefore deformations involving just the change in the function $F(u)$ is null. To obtain deformations which smooth out the singularity one should involve the transverse directions x and y, this is conveniently done in the Rosen coordinates, in these coordinates the metric in (2.1) is given by

$$ds^2 = d\mathrm{u}d\mathrm{v} + f(\mathrm{u})d\mathrm{x}^2 + g(\mathrm{u})d\mathrm{y}^2 \,. \tag{4.13}$$

The details of the transformation from the usual plane wave coordinates to the Rosen coordinates are given in (A.1) and (A.2). If the singularity in $F(u)$ is of the form $1/|u|^\alpha$ with $0 < \alpha < 1$ then for the patch containing $\mathrm{u} = 0$, the functions f and g can be chosen to be

$$f(\mathrm{u}) = \sqrt{|\mathrm{u}|} K_\nu(2\nu|\mathrm{u}|^{\frac{1}{2\nu}}) \,, \qquad g(\mathrm{u}) = \sqrt{|\mathrm{u}|} Y_\nu(2\nu|\mathrm{u}|^{\frac{1}{2\nu}}) \,. \tag{4.14}$$

We can construct a one parameter family of metrics which are not singular at $u = 0$ as follows. Consider the metric

$$ds^2 = d\mathrm{u}d\mathrm{v} + f_a^2(\mathrm{u})d\mathrm{x}^2 + g_a^2(\mathrm{u})dy^2 \,, \tag{4.15}$$

where $a \geq 0$. These are solutions of the equations of motion if f and g satisfy the following equations

$$\frac{d^2}{d\mathrm{u}^2} f_a(\mathrm{u}) = F_a(\mathrm{u}) f_a(\mathrm{u}) \,, \qquad \frac{d^2}{d\mathrm{u}^2} g_a(\mathrm{u}) = -F_a(\mathrm{u}) g(\mathrm{u}) \,, \tag{4.16}$$

where $F_a(\mathrm{u}) = (|\mathrm{u}| + a)^{-\alpha}$.[7] For $a \neq 0$, the above metric is non singular. The easiest way to see this is to go back to the plane wave coordinates of (2.1). In that coordinate system

[6] Here although $F(u) = (|u| + a)^{-\alpha}$ is dicontinuous at the origin, this is just a coordinate artifact as we will show subsequently.

[7] Note that this is not just a translation.

the function, the curvature components are given in (2.4). They are all finite at the origin as $F_a(u)$ never blows up at the origin. These components of curvature were evalutated on the parallel propagated frame along a geodesic ending at the origin, therefore they are invariants and they will remain the same for the metric given in (4.15), thus the metric is non singular. The explicit solutions for f_a and g_a in the patch containing the origin is given by

$$f_a(\mathrm{u}) = \sqrt{|\mathrm{u}| + a}K_\nu(2\nu(|\mathrm{u}| + a)^{\frac{1}{2\nu}})\,, \qquad g_a(\mathrm{u}) = \sqrt{|\mathrm{u}| + a}Y_\nu(2\nu(|\mathrm{u}| + a)^{\frac{1}{2\nu}})\,. \tag{4.17}$$

At $a = 0$ this family of metrics reduces to the singular metric characterized by $F(u) = 1/|u|^\alpha$. One mind wonder that this choice of continuation of the metic is such that its derivative is discontinuous at the orign, however we emphasize that the physical requirement for the metric to be singular is that its curvature components in the parallel propagated frame along a geodesic ending at the origin are finite, and this is certainly true for this class of metrics. In fact we now demonstrate that the discontinuity at the origin is a coordinate artifact. Consider a different choice of coordinates for $u < 0$ such that the functions f_a and g_a are given by

$$\begin{aligned} f_a(\mathrm{u}) &= \sqrt{\mathrm{u} + a}K_\nu(2\nu(\mathrm{u} + a)^{\frac{1}{2\nu}}), \qquad \text{for } u > 0 \\ f_a(\mathrm{u}) &= A\sqrt{|u| + a}K_\nu(2\nu(|u| + a)^{\frac{1}{2\nu}}) + B\sqrt{|u| + a}I_\nu(2\nu(|u| + a)^{\frac{1}{2\nu}}) \qquad \text{for } u < 0 \end{aligned} \tag{4.18}$$

here A and B are chosen so that f_a and its derivative is continuous at the orgin. Such a choice can be made because the Wronskian for the two linearly independnet solutions involved in (4.18) is given by $\pi\Gamma(1-\nu)$. Similarly one can shoose coordinates such that g_a is given by

$$\begin{aligned} g_a(\mathrm{u}) &= \sqrt{\mathrm{u} + a}Y_\nu(2\nu(\mathrm{u} + a)^{\frac{1}{2\nu}}), \qquad \text{for } u > 0\,, \\ g_a(\mathrm{u}) &= C\sqrt{|u| + a}Y_\nu(2\nu(|u| + a)^{\frac{1}{2\nu}}) + D\sqrt{|u| + a}J_\nu(2\nu(|u| + a)^{\frac{1}{2\nu}}) \qquad \text{for } u < 0 \end{aligned} \tag{4.19}$$

again C and D can be chosen such that g_a and its derivative is continuous at the origin. Thus by choosing different Rosen coordinates the discontinuity at the origin can be eliminated, therefore it is just a coordinate artifact.

We now show that the one parameter deformation constructed above is close by to the singular extended metric by evaluating the Zamolodchikov metric of the operator responsible for the deformation in the string sigma model. The string sigma model for the metric in (4.13) is given by

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(\partial_\mu U\partial^\mu U + f_a^2\partial_\mu X\partial^\mu X + g_a^2\partial_\mu Y\partial^\mu Y + \partial_\mu X^i\partial^\mu X_i\right). \tag{4.20}$$

For very small a, the deformation of the metric can be obtained by a Taylor expansion of the function $f_a(\mathrm{u})$ and $g_a(\mathrm{u})$ and retaining the first order term. Then operator which is responsible for this infinitesimal deformation in the string sigma model is given by

$$O(\tau,\sigma) = a\left(2f_a\frac{df_a}{da}\partial_\mu X\partial^\mu X + 2g_a\frac{dg_a}{da}\partial_\mu Y\partial^\mu Y\right)\Bigg|_{a=0}. \tag{4.21}$$

The Zamolodchikov metric G_{OO} or the norm of this operator is defined by

$$G_{OO}\, a^2 = \lim_{\epsilon\to 0} \epsilon^4 \langle \mathcal{O}(\tau, \sigma+\epsilon)\mathcal{O}(\tau,\sigma)\rangle\,. \tag{4.22}$$

Note that from the expansions of the Bessel functions involved in the definition of the functions f_a and g_a in (4.17) it is easy to see that the operator is well defined for $0 < \alpha < 1$. (Here we use the functions f_a and g_a from (4.17) for convenience.) To evaluate the Zamolodchikov metric it is convenient to appeal to the formula relating it to the Weil-Petersson metric in the space of metrics [31] which is given by

$$G_{OO}\, a^2 = \frac{1}{V}\int d^{26}x\sqrt{g} g^{\mu\nu} g^{\rho\sigma}\delta g_{\mu\rho}\delta g_{\nu\sigma}\,, \tag{4.23}$$

here $g_{\mu\nu}$ stands for the undeformed metric with $a=0$ given in (4.13), $\delta g_{\mu\nu}$ stands for the infinitesimal deformation in the metric by the operator (4.21) and $V = \int\sqrt{g}d^{26}x$. After substituting for these from (4.13) and (4.21) and canceling out the volume of the remaining spectator dimensions we obtain

$$G_{OO} = \frac{4}{\int \mathrm{d}u\sqrt{f^2g^2}}\int \mathrm{d}u\sqrt{f^2g^2}\left[\left(\frac{1}{f}\frac{df_a}{da}\bigg|_{a=0}\right)^2 + \left(\frac{1}{g}\frac{dg_a}{da}\bigg|_{a=0}\right)^2\right]. \tag{4.24}$$

As we are only interested in the singularity near the origin the metric in (4.13) can be taken to be that of flat space for $|u| > T$. The integrand in the numerator of above equation is supported over a compact region in u. We render the denominator finite by integrating over a large but finite region in u.[8] The only possible divergence can be at $u = 0$, in that region the functions reduce to those given in (4.14). The integrand in (4.24) contains the logarithmic derivatives of these functions and from the expansions of these function at $u = 0$ it is easy to see G_{OO} is finite for $0 < \alpha < 1$. Therefore the infinitesimal deformation in (4.21) can smooth out the singularity at $u = 0$.

Note that there are other completions for $u < 0$, for instance one can choose the following deformation $F(u) = 1/(u+a)^\alpha$, for $u > 0$ and $F(u) = 1/(|u|+a)^\beta$ for $u < 0$ and $0 < \beta < 1$. Then by solving for f_a and g_a above one can find a metric that smoothly interpolates between two different values for α from $u > 0$ and $u < 0$. It is clear that from (4.24) the metric on the moduli space for this deformation is also finite. Therefore this requirement does not single out a particular continuation of the metric to $u < 0$, thus the continuation to $u < 0$ is not unique. It will be interesting to find out if there is any other requirement from string theory which provides a unique continuation.[9]

5. Conclusions

We have seen that plane wave solutions of pure gravity with weak singularities i.e. $F(u) = 1/u^\alpha$ and $0 < \alpha < 2$, admit the sub-class with $0 < \alpha < 1$, which do not exhibit large back reaction in the presence of scalar probes. For these class of plane waves classical string

[8] Note that in [31] the Zamolodchikov metric or the Weil-Pettersson metric was evaluated for deformations in Calabi-Yau manifolds which is compact, therefore there was no need for this regularization.

[9] The author thanks the refree for emphasizing this point.

modes do not diverge at the singularity. There is a natural but not unique way to continue the metric across the singularity, this extended metric admits string propagation without the string becoming infinitely excited. This extended metric lies at a finite distacne to a family of smooth metrics in the space of metrics. The operator which smoothes out the singularity has a finite norm. The extended metric is not unique and it will be interesting to find out if there is any requirement in string theory which can single out a unique completion, certainly finiteness of string scattering amplitudes is certainly one requirement which must be studied. It will also be interesting to see if in general there are classes of weak singularities in which energies of scalar probes do not diverge and which admit smooth string propagation. The singularities studied in this paper were null singularities, and it is of interest to see if there are similar weak time like singularities. In plane waves with singularities which are supported by background Neveu-Schwarz B-fields mode creation can be suppressed due to the coupling of the B-field on the world sheet [32], here the interplay of the singular effect of the geometry is effectively canceled by and that of the B-field. This is phenomenon is potentially of interest to understand mechanisms for resolutions of time dependent singularities.

The fact that weak singularities admit space like Jacobi fields which define a finite volume element implies that they can be used to define a well behaved synchronous coordinate system near the singularity. For the plane wave analyzed in this paper these were the Rosen coordinates. One can then impose the requirement for smooth behaviour of scalar and string probes in this synchronous coordinate system and translate this to a condition on the Jacobi fields. This will provide a useful general condition for classification of metrics which admit a weak singularity.

Note added: after this work was completed the author noticed [33] which has a partial overlap with this paper.

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A. Rosen coordinates

The plane wave metric given in (2.1) is written in Brinkman coordinates which are global. Near the singularity it is convenient to use Rosen coordinates. In fact there is a choice of coordinates such that the metric is well defined at the singularity if it is weak. To convert to Rosen coordinates we perform the the following change of coordinates $(u, v, x, y) \rightarrow$

$(\mathrm{u}, \mathrm{v}, \mathrm{x}, \mathrm{y})$

$$\begin{aligned} u &= \mathrm{u}\,, \\ v &= \mathrm{v} - \frac{df}{d\mathrm{u}} f\mathrm{x}^2 - \frac{dg}{d\mathrm{u}} g\mathrm{y}^2\,, \\ x &= f\mathrm{x}\,, \\ y &= g\mathrm{y}\,, \end{aligned} \tag{A.1}$$

where f and g satisfy the equations

$$\frac{d^2 f}{d\mathrm{u}^2} = F(\mathrm{u})f\,, \frac{d^2 g}{d\mathrm{u}^2} = -F(\mathrm{u})g\,. \tag{A.2}$$

Under this coordinate transformation the plane wave metric in (2.1) reduces to

$$ds^2 = d\mathrm{u}d\mathrm{v} + f^2 d\mathrm{x}^2 + g^2 d\mathrm{y}^2\,. \tag{A.3}$$

The general solutions for f and g in (A.2) for weak singularities with $F(u) = 1/u^\alpha$ and $0 < \alpha < 2$ is given by

$$\begin{aligned} f(\mathrm{u}) &= A\sqrt{\mathrm{u}} I_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}}) + B\sqrt{\mathrm{u}} K_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}})\,, \\ g(\mathrm{u}) &= C\sqrt{\mathrm{u}} J_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}}) + D\sqrt{\mathrm{u}} Y_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}})\,, \end{aligned} \tag{A.4}$$

where $\nu = 1/(2-\alpha)$ and the functions J_ν, Y_ν and I_ν, K_ν refer to the Bessel functions and the modified Bessel functions as defined in [34]. One can obtain different Rosen coordinate systems corresponding to the various choices of the constants A, B, C and D. It is useful to write out the expansions of the functions appearing in (A.4) around the origin, for the functions involving I_ν and J_ν the expansion are given by

$$\begin{aligned} \sqrt{\mathrm{u}} I_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}}) &= \nu^\nu \mathrm{u} \left(\frac{1}{\Gamma(\nu+1)} + \frac{\nu^2 \mathrm{u}^{2-\alpha}}{1!\Gamma(\nu+2)} + \frac{\nu^4 \mathrm{u}^{2(2-\alpha)}}{2!\Gamma(\nu+3)} + \cdots \right), \\ \sqrt{\mathrm{u}} J_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}}) &= \nu^\nu \mathrm{u} \left(\frac{1}{\Gamma(\nu+1)} - \frac{\nu^2 \mathrm{u}^{2-\alpha}}{1!\Gamma(\nu+2)} + \frac{\nu^4 \mathrm{u}^{2(2-\alpha)}}{2!\Gamma(\nu+3)} - \cdots \right). \end{aligned} \tag{A.5}$$

For the functions involving K_ν and Y_ν and when ν is not an integer the expansions are given by

$$\begin{aligned} \sqrt{\mathrm{u}} K_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}}) &= \frac{\pi}{2\sin(\nu\pi)} \nu^{-\nu} \left(\frac{1}{\Gamma(-\nu+1)} + \frac{\nu^2 \mathrm{u}^{2-\alpha}}{1!\Gamma(-\nu+2)} + \frac{\nu^4 \mathrm{u}^{2(2-\alpha}}{2!\Gamma(-\nu+3)} + \cdots \right) - \\ &\quad - \frac{\pi}{2\sin(\nu\pi)} \sqrt{\mathrm{u}} I_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}}), \\ \sqrt{\mathrm{u}} Y_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}}) &= \frac{-1}{\sin(\nu\pi)} \nu^{-\nu} \left(\frac{1}{\Gamma(-\nu+1)} - \frac{\nu^2 \mathrm{u}^{2-\alpha}}{1!\Gamma(-\nu+2)} + \frac{\nu^4 \mathrm{u}^{2(2-\alpha)}}{2!\Gamma(-\nu+3)} - \cdots \right) + \\ &\quad + \cot(\nu\pi) \sqrt{\mathrm{u}} J_\nu(2\nu \mathrm{u}^{\frac{1}{2\nu}})\,. \end{aligned} \tag{A.6}$$

Note that the above functions are finite at the origin. When $1 \leq \alpha < 2$ there is a possibility of ν being an integer, then there are logarithmic terms in the expansion for K_ν and J_ν,

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still these functions are finite at the origin, the logarithmic terms and the first few terms in the expansion of K_n is given by

$$\begin{aligned}\sqrt{\mathrm{u}}K_n(2n\mathrm{u}^{\frac{1}{2n}}) &= (-1)^{n+1}\sqrt{u}I_n(2n\mathrm{u}^{\frac{1}{2n}})\ln(n\mathrm{u}^{\frac{1}{2n}}) + \\ &+\frac{n^{-n}}{2}\left((n-1)! - \frac{(n-2)!n^2\mathrm{u}^{(2-\alpha)}}{2!} + \cdots (-1)^{n-1}\frac{n^{2(n-2)}\mathrm{u}^{(n-1)(2-\alpha)}}{(n-1)!}\right) + \\ &+(-1)^n\frac{n^n}{2}\mathrm{u}\left(\frac{(\psi(1)+\psi(n+1))}{n!} + \frac{(\psi(2)+\psi(n+2))n^2\mathrm{u}^{2-\alpha}}{2!(n+2)!} + \cdots\right),\end{aligned} \tag{A.7}$$

here $n = 1/(2-\alpha)$ is an integer and ψ is the digamma function defined [34]. Similarly the expansion for Y_n is given by

$$\begin{aligned}\sqrt{\mathrm{u}}Y_n(2n\mathrm{u}^{\frac{1}{2n}}) &= \frac{2}{\pi}\sqrt{u}J_n(2n\mathrm{u}^{\frac{1}{2n}})\ln(n\mathrm{u}^{\frac{1}{2n}}) - \\ &-\frac{n^{-n}}{\pi}\left((n-1)! + \frac{(n-2)!n^2\mathrm{u}^{(2-\alpha)}}{2!} + \cdots \frac{n^{2(n-2)}\mathrm{u}^{(n-1)(2-\alpha)}}{(n-1)!}\right) - \\ &-\frac{n^n}{\pi}\mathrm{u}\left(\frac{(\psi(1)+\psi(n+1))}{n!} - \frac{(\psi(2)+\psi(n+2))n^2\mathrm{u}^{2-\alpha}}{2!(n+2)!} + \cdots\right).\end{aligned} \tag{A.8}$$

From the above expressions in (A.6), (A.7) and (A.8) it is clear that for $0 < \alpha < 2$ the functions in (A.4) involving K_ν and Y_ν are finite at the origin and from (A.5) the functions in (A.4) involving I_ν and J_ν vanish at the origin. The jacobian for transformation to the Rosen coordinates is given by the product fg, therefore to have a well defined coordinate system in the patch which contains $u = 0$, B and D is not zero. It is convenient to choose $A = B = 0$ in the Rosen coordinates in this patch.

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